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Improving the G. E. Hutchinson Population Model for Functional-Differential Equations

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Abstract: The article is devoted to improve the G. E. Hutchinson population model for functio-nal-differential equations. The functional-differential equation with maxima is considered under the boundary conditions. The issues of unique classical solvability and construction of a solution to a boundary problem for a nonhomogeneous functional-differential equation containing maxima of unknown function are studied. The proposed functional-differential equation with maxima is considered as a mathematical model of population dynamics of a species. Sufficient coefficient conditions for unique classical solvability of the boundary problem are established. The method of successive approximations is used with combination of the contracting mapping method. The examples with illustrations of the properties of solution of the functional-differential equation with maxima are given.

Keywords and phrases: G. E. Hutchinson population model, boundary problem, functional-differential equation, maxima of unknown function, classical solvability.

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1 Introduction. Statement of the problem

In many studies, differential equations are a tool for mathematical modeling of processes and studying their properties (see, for example, [2]–[9] and see also [1]–[18]). Mathematical modelling in ecology has a long history leading back to the famous works of Malthus [12], Quetelet [15] and Verhulst [17] on principals of demographic regulation. These early models articulated fundamental problems in population dynamics that have been addressed in much later works. In the first part of the 20th century, the classical works

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of Lotka [11] and Volterra [18] on dynamics of interacting species radically transformed the entire idea of modelling in ecology. Their works, along with many other important contributions, eventually resulted in the emergence of theoretical ecology as a new science. For a short history of the implementation of mathematics in ecology, see [13]. Note that the questions on the main mechanisms of populations regulation addressed by Lotka and Volterra as well as by the earliest modelers Malthus, Quetelet and Verhulst are still in the focus of theoretical ecology. In particular, the following mathematical model was proposed to describe the population dynamics of a species ([19], [16])

$$\dot{x}(t) + r x(t) + r x^2(t) = 0, \quad t \geq 0, \tag{1}$$

where $x(t) = N(t) + K$, $N(t)$ is the population size of the species, K is the capacity of the habitat, r is the reproduction coefficient.

Equation (1) with initial data $x(0) = x_0$ has the following properties:

1). $N(t) > 0$ for $x_0(t) > 0$;

2). $\lim\limits_{t\to\infty} N(t) = K$;

3). The function $N(t)$ is increasing for $0 < x_0 < K$, and is decreasing for $x_0 < K$;

4). The equilibrium position $N(t) \equiv K$ is globally asymptotically stable.

Equation (1) describes well the dynamics of population growth of protozoan microorganisms. In this regard, G. E. Hutchinson (1948) proposed a model described by a differential equation with delay [9], [1]

$$\dot{x}(t) + rx(t-h) + rx(t-h)x(t) = 0, \quad t \geq 0, \tag{2}$$

where $x(t) = N(t) + K$, h characterizes the average reproductive age of the species.

Hutchinson used equation (2) to describe changes in the size of a herd of cows on a pasture. However, equation (2) has been used by other scientists to describe other processes. Let us consider equation (2) with the initial condition $x(t) = \varphi(t)$, $t \in [-h, 0]$, $x(0) = \varphi_0$. For small $r > 0$ the equation (2) has a stability region. Asymptotic stability of the equilibrium position occurs when $N(t) \equiv K$, $t \geq 0$, $0 < rh < \pi/2$. It should be noted that periodic fluctuations appear in the mathematical model for $rh > \pi/2$. Consequently, in Hutchinson's model, the reproductive rate r and the average reproductive age h of the species play a fundamental role.

In this paper we propose the following model to describe the population dynamics of the species

$$\dot{x}(t) + r \max_{\tau\in[t-h,t]} x(\tau) + r x(t) \max_{\tau\in[t-h,t]} x(\tau) = 0, \; t \geq 0, \tag{3}$$

where $x(t) = N(t) + K$, τ is a variable value that provides the maximum number of species on the segment $[t-h, t]$, h characterizes the size of the reproductive age of the species, r is reproduction rate.

Theoretically, differential equations are considered, particularly, in [7]–[24].

If in model (3) we take into account the heterogeneous habitat, migration factors associated with the amount of available food, and the influence of predators, then the mathematical model takes on a more complex form

$$\dot{x}(t)+r\max_{\tau\in[t-h,t]}x(\tau)+rx(t)\max_{\tau\in[t-h,t]}x(\tau)=f(t,x(t)),\ t\geq 0, \tag{4}$$

where f is an external disturbing influence on the population dynamics of a species.

We will consider equation (4) with the following initial conditions

$$x(t)=\varphi(t),\ \ t\in[-h,0],\ \ x(0)+x(T)=\varphi_0. \tag{5}$$

We integrate the initial value problem (4), (5) on the interval $(0,t)$:

$$x(t)=x(0)+\int_0^t\Big[-r\max_{\tau\in[s-h,s]}x(\tau)-rx(s)\max_{\tau\in[s-h,s]}x(\tau)+f(s,x(s))\Big]ds. \tag{6}$$

Taking into account conditions (5), from equation (6) we obtain

$$x(0)=\frac{\varphi_0}{2}-\frac{1}{2}\int_0^T\Big[-r\max_{\tau\in[s-h,s]}x(\tau)-rx(s)\max_{\tau\in[s-h,s]}x(\tau)+f(s,x(s))\Big]ds. \tag{7}$$

Substituting (7) into equation (6), we obtain the nonlinear Fredholm integral equation

$$x(t)=J(t;x(t))\equiv\frac{\varphi_0}{2}+$$

$$+\frac{K_0}{2}\int_0^T\Big[-r\max_{\tau\in[s-h,s]}x(\tau)-rx(s)\max_{\tau\in[s-h,s]}x(\tau)+f(s,x(s))\Big]ds, \tag{8}$$

where

$$K_0=\begin{cases}-\frac{1}{2},\ t\leq s\leq T,\\ \frac{1}{2},\ 0\leq s<t.\end{cases}$$

2 Unique solvability

We use a Banach space $C\big([0,T],\mathbb{R}\big)$, that contains a function $u(t)$, defined and continuous on a segment $[0,T]$. These space $C\big([0,T],\mathbb{R}\big)$ we consider by the following norm

$$\|\,u(t)\,\|_{C[0,T]}=\max_{0\leq t\leq T}|\,u(t)\,|\,.$$

Let us also consider the another Banach space

$$BD\left([0,T],\mathbb{R}\right)=\Big\{u:[0,T]\to\mathbb{R};\ u(t)\in C\left([0,T],\mathbb{R}\right)\Big\},$$

in which the function exists and is bounded by the norm

$$\|\,u(t)\,\|_{BD[0,T]}=\|\,u(t)\,\|_{C[0,T]}+h\,\|\,\dot{u}(t)\,\|_{C[0,T]}\,,$$

where $0<h=\text{const}$.

Lemma 1 ([7, 20, 21]). *For the difference of two functions with maxima there holds the following estimate*

$$\left\| \max_{\tau\in[t-h,t]} x(\tau) - \max_{\tau\in[t-h,t]} y(\tau) \right\|_{C[0,T]} \leq \| x(t) - y(t) \|_{C[0,T]} + h \| \dot{x}(t) - \dot{y}(t) \|_{C[0,T]}.$$

Theorem 1. *Let the following conditions be fulfilled:*
1) $M_f = \| f(t,x) \|_{C[0,T]}, \ 0 < M_f = \text{const}$;
2) $| f(t,x_1) - f(t,x_2) | \leq L_f | x_1 - x_2 |, \ 0 < L_f = \text{const}$;
3) $\rho_i = \max\left\{\rho_{i1}; h\rho_{i2}\right\} < 1, \ i = 1,2, ..., n-1.$

Then the equation (8) has a unique solution in the class $BD([0,T],\mathbb{R})$, which can be found by following iteration process:

$$\left[\begin{array}{l} x_0(t) = \frac{\varphi_0}{2}, \\ x_n(t) = J(t; x_{n-1}(t)), \ n = 1,2,3,... \end{array} \right. \tag{9}$$

Proof. For the first approximation from (8) and (9) we have

$$\| x_1(t) \| \leq \frac{|\varphi_0|}{2} +$$

$$+ \frac{1}{4}\int_0^T \left[|r| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + |r| \| x_0(s) \| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + \| f(s, x_0(s)) \| \right] ds \leq$$

$$\leq \frac{|\varphi_0|}{2} + \frac{T}{4}\left[|r| \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + |r| \frac{|\varphi_0|}{2} \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + M_f \right] =$$

$$= \frac{|\varphi_0|}{2} + \rho_{01} < \infty, \tag{10}$$

where

$$\rho_{01} = \frac{T}{4}\left[|r| \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + |r| \frac{|\varphi_0|}{2} \max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + M_f \right].$$

Taking into account estimate (10) from (9) we have estimates

$$\| x_1(t) - x_0(t) \| \leq$$

$$\leq \frac{1}{4}\int_0^T \left[|r| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + |r| \| x_0(s) \| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + \| f(s, x_0(s)) \| \right] ds \leq \rho_{01}. \tag{11}$$

Analogously, from (4) we have

$$\| \dot{x}_1(t) - \dot{x}_0(t) \| \leq$$

$$\leq |r| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + |r| \| x_0(s) \| \left\| \max_{\tau\in[s-h,s]} x_0(\tau) \right\| + \| f(s, x_0(s)) \| \leq$$

$$\leq |r| \max\left\{ \|\varphi(t)\|, \frac{|\varphi_0|}{2} \right\} + |r| \frac{|\varphi_0|}{2} \max\left\{ \|\varphi(t)\|, \frac{|\varphi_0|}{2} \right\} + M_f = \rho_{02}. \qquad (12)$$

From (11) and (12) we obtain that there holds the estimate

$$\|x_1(t) - x_0(t)\|_{BD[0,T]} \leq \rho_{01} + h\rho_{02}.$$

For the second difference we have

$$\|x_2(t) - x_1(t)\| \leq \frac{T|r|}{4} \left\| \max_{\tau\in[t-h,t]} x_1(\tau) - \max_{\tau\in[t-h,t]} x_0(\tau) \right\| +$$

$$+ \frac{T|r|}{4} \|x_1(t)\| \left\| \max_{\tau\in[t-h,t]} x_1(\tau) - \max_{\tau\in[t-h,t]} x_0(\tau) \right\| +$$

$$+ \frac{T|r|}{4} \|x_1(t) - x_0(t)\| \left\| \max_{\tau\in[t-h,t]} x_0(\tau) \right\| + \frac{T}{4} \|f(t, x_1(t)) - f(t, x_0(t))\|.$$

We apply the lemma to the last inequality:

$$\|x_2(t) - x_1(t)\| \leq \frac{T|r|}{4} \Big(\|x_1(t) - x_0(t)\| + h\|\dot{x}_1(t) - \dot{x}_0(t)\| \Big) +$$

$$+ \frac{T|r|}{4} \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) \Big(\|x_1(t) - x_0(t)\| + h\|\dot{x}_1(t) - \dot{x}_0(t)\| \Big) +$$

$$+ \frac{T|r|}{4} \|x_1(t) - x_0(t)\| \max\left\{ \|\varphi(t)\|, \frac{|\varphi_0|}{2} \right\} + \frac{T}{4} L_f \|x_1(t) - x_0(t)\|.$$

Hence, we obtain

$$\|x_2(t) - x_1(t)\| \leq \chi_{11} \|x_1(t) - x_0(t)\| + \chi_{12} h \|\dot{x}_1(t) - \dot{x}_0(t)\|, \qquad (13)$$

where

$$\chi_{11} = \frac{T}{4} \left[|r| \left(1 + \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) + \max\left\{ \|\varphi(t)\|, \frac{|\varphi_0|}{2} \right\} \right) + L_f \right],$$

$$\chi_{12} = \frac{T|r|}{4} \left(1 + \frac{|\varphi_0|}{2} + \rho_{01} \right).$$

Since $\chi_{11} \geq \chi_{12}$, the estimate (13) we rewrite as

$$\|x_2(t) - x_1(t)\| \leq \rho_{11} \Big(\|x_1(t) - x_0(t)\| + h\|\dot{x}_1(t) - \dot{x}_0(t)\| \Big) \leq$$

$$\leq \rho_{11} (\rho_{01} + h\rho_{02}), \qquad (14)$$

where $\rho_{11} = \chi_{11}$.

Analogously we have

$$\|\dot{x}_2(t) - \dot{x}_1(t)\| \leq |r| \Big(\|x_1(t) - x_0(t)\| + h\|\dot{x}_1(t) - \dot{x}_0(t)\| \Big) +$$

$$+|r|\left(\frac{|\varphi_0|}{2}+\rho_{01}\right)\Big(\|x_1(t)-x_0(t)\|+h\,\|\dot{x}_1(t)-\dot{x}_0(t)\|\Big)+$$

$$+|r|\,\|x_1(t)-x_0(t)\|\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}+L_f\,\|x_1(t)-x_0(t)\|\leq$$

$$\leq\chi_{21}\,\|x_1(t)-x_0(t)\|+\chi_{22}h\,\|\dot{x}_1(t)-\dot{x}_0(t)\|,$$

$$\chi_{21}=|r|\left(1+\left(\frac{|\varphi_0|}{2}+\rho_{01}\right)+\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}\right)+L_f,$$

$$\chi_{22}=|r|\left(1+\frac{|\varphi_0|}{2}+\rho_{01}\right).$$

From last estimate we get

$$\|\dot{x}_2(t)-\dot{x}_1(t)\|_{BD[0,T]}\leq\rho_{12}\Big(\|x_1(t)-x_0(t)\|+h\,\|\dot{x}_1(t)-\dot{x}_0(t)\|\Big)\leq$$

$$\leq\rho_{12}\,(\rho_{01}+h\,\rho_{02}),\tag{15}$$

where $\rho_{12}=\chi_{21}$.

From (14) and (15) we obtain that there hold the estimates

$$\|x_2(t)-x_1(t)\|_{BD[0,T]}\leq\rho_1\,\|x_1(t)-x_0(t)\|_{BD[0,T]},$$

where $\rho_1=\max\{\rho_{11};h\rho_{12}\}<1$;

$$\|x_2(t)-x_1(t)\|_{BD[0,T]}\leq\rho_{11}\,(\rho_{01}+h\,\rho_{02})+h\,\rho_{12}\big(\rho_{01}+h\,\rho_{02}\big)=$$

$$=\big(\rho_{11}+h\,\rho_{12}\big)\big(\rho_{01}+h\,\rho_{02}\big).\tag{16}$$

We need the following estimate

$$\|x_2(t)\|\leq\frac{|\varphi_0|}{2}+$$

$$+\frac{1}{4}\int_0^T\left[|r|\left\|\max_{\tau\in[s-h,s]}x_1(\tau)\right\|+|r|\,\|x_1(s)\|\left\|\max_{\tau\in[s-h,s]}x_1(\tau)\right\|+\|f(s,x_1(s))\|\right]ds\leq$$

$$\leq\frac{|\varphi_0|}{2}+\frac{T}{4}\left[|r|\left(\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}+\rho_{01}\right)+\right.$$

$$\left.+|r|\left(\frac{|\varphi_0|}{2}+\rho_{01}\right)\left(\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}+\rho_{01}\right)+M_f\right]=$$

$$\leq\frac{|\varphi_0|}{2}+\frac{T}{4}\left[|r|\left(\max\left\{\|\varphi(t)\|,\frac{|\varphi_0|}{2}\right\}+\rho_{01}\right)\left(1+\frac{|\varphi_0|}{2}+\rho_{01}\right)+M_f\right]<\infty.\tag{17}$$

Taking into account (17), analogously to (15) and (16) we consider next difference

$$\|x_3(t)-x_2(t)\|\leq\frac{T\,|r|}{4}\left\|\max_{\tau\in[t-h,t]}x_2(\tau)-\max_{\tau\in[t-h,t]}x_1(\tau)\right\|+$$

$$+\frac{T\,|\,r\,|}{4}\,\|\,x_2(t)\,\|\left\|\max_{\tau\in[t-h,t]}x_2(\tau)-\max_{\tau\in[t-h,t]}x_1(\tau)\right\|+$$

$$+\frac{T\,|\,r\,|}{4}\,\|\,x_2(t)-x_1(t)\,\|\left\|\max_{\tau\in[t-h,t]}x_1(\tau)\right\|+\frac{T}{4}\,\|\,f(t,x_2(t))-f(t,x_1(t))\,\|\leq$$

$$\leq\rho_{21}\Big(\,\|\,x_2(t)-x_1(t)\,\|+h\,\|\,\dot{x}_2(t)-\dot{x}_1(t)\,\|\,\Big),\tag{18}$$

where we denoted

$$\rho_{21}=\frac{T}{4}\left[|\,r\,|\left(\|\,x_2(t)\,\|+\left\|\max_{\tau\in[t-h,t]}x_1(\tau)\right\|\right)+L_f\right].$$

Similarly, from (15) and (16) we obtain

$$\|\,\dot{x}_3(t)-\dot{x}_2(t)\,\|\leq\rho_{22}\Big(\,\|\,x_2(t)-x_1(t)\,\|+h\,\|\,\dot{x}_2(t)-\dot{x}_1(t)\,\|\,\Big),\tag{19}$$

where

$$\rho_{22}=|\,r\,|\left(\|\,x_2(t)\,\|+\left\|\max_{\tau\in[t-h,t]}x_1(\tau)\right\|\right)+L_f.$$

From the estimates (18) and (19) we obtain that there hold the estimates

$$\|\,x_3(t)-x_2(t)\,\|_{BD[0,T]}\leq\rho_2\,\|\,x_2(t)-x_1(t)\,\|_{BD[0,T]}\,,$$

where $\rho_2=\max\big\{\rho_{21};h\rho_{22}\big\}<1$;

$$\|\,x_3(t)-x_2(t)\,\|_{BD[0,T]}\leq\rho_{21}\big(\rho_{11}+h\,\rho_{12}\big)\big(\rho_{01}+h\,\rho_{02}\big)+h\,\rho_{22}=$$

$$=\big(\rho_{21}+h\,\rho_{22}\big)\big(\rho_{11}+h\,\rho_{12}\big)\big(\rho_{01}+h\,\rho_{02}\big).\tag{20}$$

Continuing the process (20), we obtain the estimates for arbitrary n:

$$\|\,x_n(t)-x_{n-1}(t)\,\|_{BD[0,T]}\leq\rho_{n-1}\,\|\,x_{n-1}(t)-x_{n-2}(t)\,\|_{BD[0,T]}\,,$$

where $\rho_{n-1}=\max\big\{\rho_{(n-1)1};h\rho_{(n-1)2}\big\}<1$;

$$\|\,x_n(t)-x_{n-1}(t)\,\|_{BD[0,T]}\leq$$

$$\leq\big(\rho_{(n-1)1}+h\,\rho_{(n-1)2}\big)\cdots\big(\rho_{21}+h\,\rho_{22}\big)\big(\rho_{11}+h\,\rho_{12}\big)\big(\rho_{01}+h\,\rho_{02}\big),\tag{21}$$

where

$$\rho_{(n-1)1}=\frac{T}{4}\left[|\,r\,|\left(\|\,x_{n-1}(t)\,\|+\left\|\max_{\tau\in[t-h,t]}x_{n-2}(\tau)\right\|\right)+L_f\right],$$

$$\rho_{(n-1)2}=|\,r\,|\left(\|\,x_{n-1}(t)\,\|+\left\|\max_{\tau\in[t-h,t]}x_{n-2}(\tau)\right\|\right)+L_f.$$

From (21) we obtain the following estimate:

$$\| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} \le \rho_0\rho_1\rho_2 \cdot \ldots \cdot \rho_{n-2}\rho_{n-1},$$

where $\rho_i = \max\left\{\rho_{i1}; h\rho_{i2}\right\} < 1,\ i = 1, 2, \ldots, n-1.$

The quantities r, h, L_f we choose such that from (21) implies

$$\lim_{n\to\infty} \| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} = 0.$$

Hence, we deduce that the right-hand side of the equation (8) is contracting operator and the problem (4), (5) has a unique solution in the space $BD\big([0,T],\mathbb{R}\big)$. □

3 Examples

The monotone increasing solutions of the equation (3) coincide with the increasing solutions of the differential equations (1).

Example 1. We consider the equation

$$x'(t) + 0.5 \max_{\tau\in[t-1,t]} x(\tau) + 0.5x(t) \max_{\tau\in[t-1,t]} x(\tau) = 2t + (1 + 0.5t^2)x(t),\ \ t \ge 0. \tag{22}$$

We will consider equation (22) with the following initial condition

$$x(t) = \varphi(t) = t + 1,\ \ t \in [-1, 0],\ \ x(0) = 1. \tag{23}$$

The solution $x(t) = 1 + t^2$ of the problem (22), (23) coincides with the solution of the equation

$$x'(t) + 0.5x(t) + 0.5x^2(t) = 2t + (1 + 0.5t^2)x(t),\ \ t \ge 0$$

with condition (23).

So, the monotone increasing solutions of the equation (3) coincide with the increasing solutions of the differential equations (1).

Now we consider the monotone decreasing solutions of the equation (3), which coincide with the decreasing solutions of the differential equations (2).

Example 2. We have to consider the equation

$$x'(t) + 0.2 \max_{\tau\in[t-1,t]} x(\tau) + 0.2x(t) \max_{\tau\in[t-1,t]} x(\tau) = \frac{0.2(t^2+1)}{t+1} x^2(t),\ \ t \ge 0. \tag{24}$$

We will solve the equation (24) with the following initial condition

$$x(t) = \varphi(t) = \frac{1}{0.5t+2},\ \ t \in [-1, 0],\ \ x(0) = 1. \tag{25}$$

The solution

$$x(t) = \frac{1}{t+2},$$

of the problem (24), (25), coincides with the solution of the equation

$$x'(t)+0.5x(t-1)+0.5x(t)x(t-1)=2t+(1+0.5t^2)x(t),\ \ t\geq 0.$$

Example 3. If the solution of the differential equations with maxima (3) or (4) is not monotone, then the solution could not coincide with the solution of any differential equations. For examples, we consider in the coordinate system the following functions, which are different:

1) $x(t)=\sin t,\ \ t\in[0,2\pi]$;
2) $x(t)=\sin\left(t-\frac{\pi}{6}\right),\ \ t\in[0,2\pi]$;
3) $x(t)=\max\limits_{\tau\in[t-(\pi/6),t]}\sin(\tau),\ \ t\in[0,2\pi]$.

Figure 1 shows the difference between function with maxima and corresponding functions without any maxima.

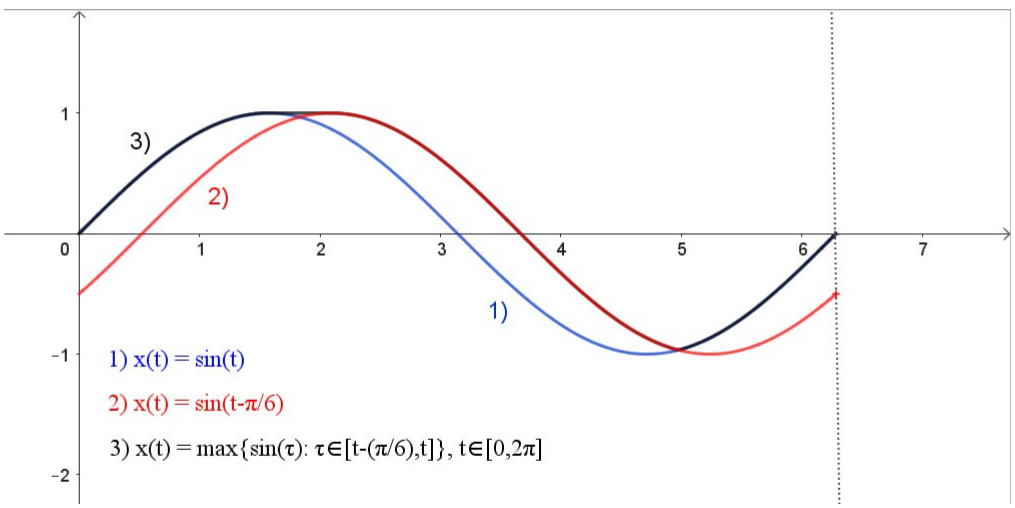

Figure 1. Difference between functions with maxima and without any maxima

Example 4. We consider on a coordinate system the resonance for following functions:

1) $x(t)=t\sin t,\ \ t\in[0,2\pi]$;
2) $x(t)=t\sin\left(t-\frac{\pi}{6}\right),\ \ t\in[0,2\pi]$;
3) $x(t)=t\max\limits_{\tau\in[t-(\pi/6),t]}\sin(\tau),\ \ t\in[0,2\pi]$.

Figure 2 shows the influence of a maxima to the resonance phenomenon.

Example 5. Now we consider the influence of delay to behavior of given functions:

1) $x(t)=\max\limits_{\tau\in[t-(\pi/6),t]}\sin(\tau),\ \ t\in[0,2\pi]$;
2) $x(t)=\max\limits_{\tau\in[t-(\pi/2),t]}\sin(\tau),\ \ t\in[0,2\pi]$;
3) $x(t)=\max\limits_{\tau\in[t-\pi,t]}\sin(\tau),\ \ t\in[0,2\pi]$.

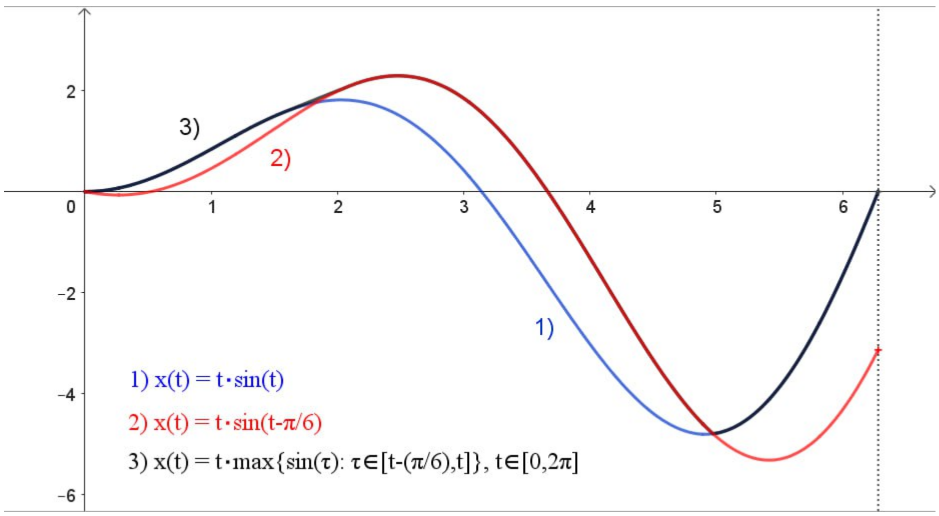

Figure 2. Influence of a maxima to the resonance phenomenon

Figure 3 shows the influence of the lag size on the behavior of a function with maxima. As the delay increases, the direct line part of the function with maxima increases, i.e., the reproductive age of the species stabilizes on the large part of the interval.

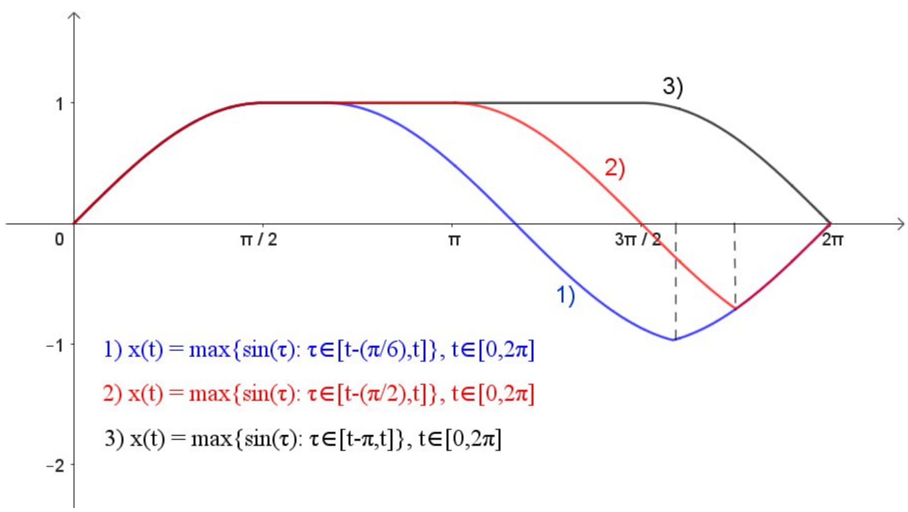

Figure 3. Influence of the lag size on the behavior of a function

Example 6. We study the role of reproduction rate r to solution of the equation with maxima. We consider on a coordinate system the following differential equations with maxima with initial function $x(t)=t^2$, $t\in[-1,0]$:

1) $x'(t)-2(1+x(t))\max\limits_{\tau\in[t-1,t]} x(\tau)=0,\ \ t\geq 0;$

2) $x'(t)+0.1(1+x(t))\max_{\tau\in[t-1,t]} x(\tau)=0,\ t\geq 0;$
3) $x'(t)+2(1+x(t))\max_{\tau\in[t-1,t]} x(\tau)=0,\ t\geq 0;$
4) $x'(t)+6(1+x(t))\max_{\tau\in[t-1,t]} x(\tau)=0,\ t\geq 0.$

Figure 4 shows the role of reproduction rate r to solution of the equation with maxima. In particular, we can see that the reproduction rate r is not a negative number. Moreover,

$$\lim_{r\to 0} x(t)=0,$$

as well. Consequently, the differential equation (3) has no monotone increasing solution. So, the population dynamics of a species is described as the oscillatory process.

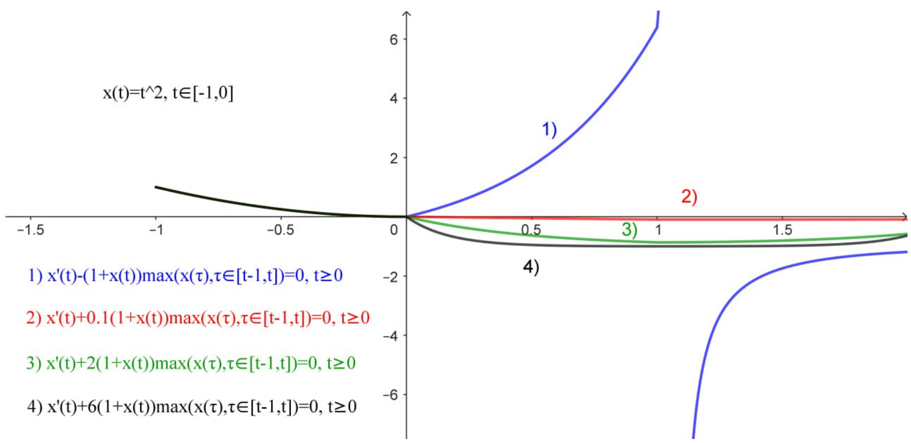

Figure 4. Role of reproduction rate r

Example 7. Solve the differential equation $x'(t)+2(1+x(t))\max_{\tau\in[t-2,t]} x(\tau)=0,\ t\geq 0.$ We consider the equation on a coordinate system with the different initial functions:

1) $x(t)=t^2,\ t\in[-2,0];$
2) $x(t)=t^3+t^2,\ t\in[-2,0];$
3) $x(t)\frac{t^2+2t}{t^3+t^2+1},\ t\in[-2,0];$
4) $x(t)=\sin t^2,\ t\in[-2,0].$

Figure 5 shows the influence of the initial functions to the behavior of solution of the differential equation with maxima.

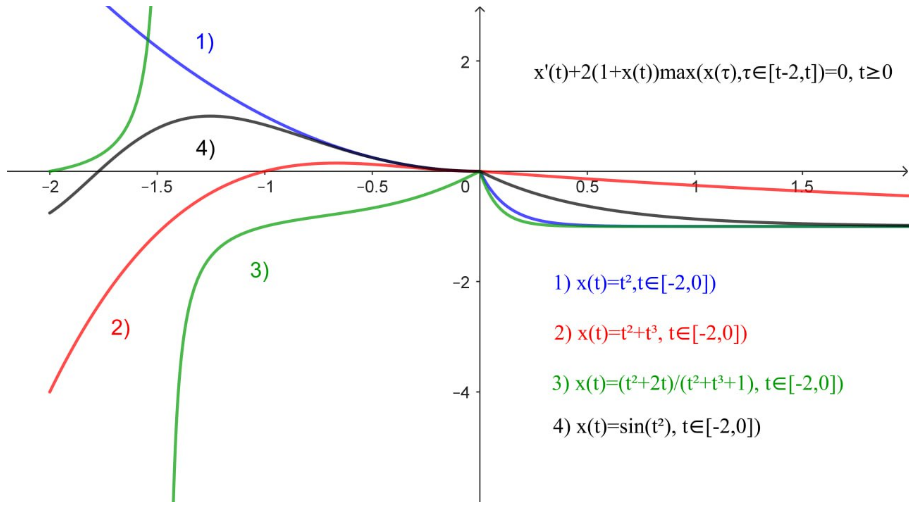

Figure 5. Influence of the initial functions to the behavior of solution

4 Conclusions

The article is devoted to improve the G. E. Hutchinson population model for functional-differential equations. The functional-differential equation with maxima is considered under the boundary conditions. The issues of unique classical solvability and construction of a solution to a boundary problem for a nonhomogeneous functional-differential equation, containing maxima of unknown function, are studied. The proposed functional-differential equation with maxima is considered as a mathematical model of population dynamics of a species. Sufficient coefficient conditions for unique classical solvability of the boundary problem are established. The method of successive approximations is used with combination of the contracting mapping method. The examples with illustrations of the properties of functions and the properties of solutions of the functional-differential equation with maxima are given.

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